MATH2050B 1920 HW3

TA's solutions to selected problems

Q1. Let a, b be positive real numbers. Show that $a < b$ iff $a^2 < b^2$.

Solution. $a < b$ iff $0 < b - a$ iff $(b + a)0 < (b + a)(b - a)$ (i.e. $0 < b^2 - a^2$) iff $a^2 < b^2$.

Q2. Let b, c be real numbers. Show that the quadratic equation $x^2 - 2bx + c = 0$ is solvable in $\mathbb{R}$ iff $b^2 - c$ is non-negative.

Solution. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2 - 2bx + c = (x - b)^2 + c - b^2$. $x \in \mathbb{R}$ is a root to f iff $(x - b)^2 = b^2 - c$.

If such a solution x exists in $\mathbb{R}$, then $b^2 - c = (x - b)^2 \geq 0$. Conversely if $b^2 - c \geq 0$, then there is a real number(the square root) y with $y^2 = b^2 - c$. Then $x = y + b$ is a solution to the problem $f(x) = 0$.

Q3. Let z, a be positive real numbers such that $z^2 > a$. Show that there exists a natural number m such that $(1/m) < z$ and $(z - 1/m)^2 > a$.

Solution. Because $z > 0$, therefore $1/z > 0$. By Archimedean Principle, there is m_z for which $m_z > 1/z$. Clearly $m > 1/z$ for all $m \geq m_1$, so that

$$z > \frac{1}{m}, \qquad \forall m \geq m_1.$$

Because $z^2 - a > 0$, let $\delta = z^2 - a > 0$. By Archimedean Principle, there is m_2 such that $m_2 > 2z/\delta$. Now let m be a natural number greater than m_1 and m_2, then $z > 1/m$. And $\delta > 2z/m$ so that:

$$\begin{aligned}
(z - \frac{1}{m})^2 - a &= (z - \frac{1}{m})^2 - z^2 + z^2 - a \\
&= (\frac{1}{m})(\frac{1}{m} - 2z) + z^2 - a \\
&= (\frac{1}{m})(\frac{1}{m} - 2z) + \delta \\
&= \frac{1}{m^2} - \frac{2z}{m} + \delta \\
&> -\frac{2z}{m} + \delta > 0.
\end{aligned}$$

Q4. Let z, a be positive such that $z^2 < a$. Show that there exists a natural number n such that $(z + 1/n)^2 < a$.

Solution. As $a - z^2 > 0$, let $\delta = a - z^2 > 0$. By Archimedean Principle there is n such that $n > (2z + 1)/\delta$. This implies $\delta - (2z + 1)/n > 0$ and so for this n we have:

$$\begin{aligned}
a - (z + \frac{1}{n})^2 &= a - z^2 + z^2 - (z + \frac{1}{n})^2 \\
&= \delta - \frac{2z}{n} - \frac{1}{n^2} \\
&\geq \delta - \frac{2z}{n} - \frac{1}{n} && (-\tfrac{1}{n^2} \geq -\tfrac{1}{n}) \\
&= \delta - \frac{1}{n}(2z + 1) > 0.
\end{aligned}$$

Q5. Let a be positive and $B = \{x \in \mathbb{R} : x > 0, x^2 > a\}$. Show that B is non-empty, $z := \inf B$ exists in $\mathbb{R}$. Show further that $z^2 = a$.

Solution. B is non-empty because $a+1 \in B$. B is bounded below by 0 and so $z = \inf B$ exists in $\mathbb{R}$. Since 0 is a lower bound, so $z \geq 0$. And z must be positive because if $z = 0$, then there would be some $x \in B$ with $x < 1, x < a$, so that $x^2 < 1 \cdot a$, which is a contradiction.

Now suppose on the contrary that $z^2 \neq a$, then either $z^2 > a$ or $z^2 < a$.

If $z^2 > a$, then (by **Q3**) there is some m with $z - 1/m > 0$ $(z - \frac{1}{m})^2 > a$. So $z - 1/m \in B$. Contradiction.

If $z^2 < a$, then (by **Q4**) there is some m with $(z + 1/m)^2 < a$. Then $z + 1/m$ is a lower bound for B strictly bigger then z. Contradiction.

So $z^2 = a$.

Q6. Let $a > 0$. Similar to **Q5** but use suitable set A and $\sup A$ to show the existence of the positive square root of a.

Solution. $A = \{x \in \mathbb{R} : x > 0, x^2 < a\}$. Because a is finite so A is bounded above. (Reason: or else there must be some $x \in A$ so that $x > a + 1$. Then $x^2 \geq (a+1)^2 > a$)

So $z := \sup A$ exists, similar to **Q5** we can show $z^2 = a$.

Q7. Show $||x| - |y|| \leq |x - y| \leq |x| + |y|$ for all $x, y \in \mathbb{R}$ and describe when the inequality is strict. Sketch/interpret your results geometrically.

Solution. The absolute value satisfies $|x + y| \leq |x| + |y|$ for all real x, y and $|\lambda x| = |\lambda| \cdot |x|$ for all real λ, x.

Since $|x| = |x - y + y| \leq |x - y| + |y|$, so $|x| - |y| \leq |x - y|$. Similarly $|y| \leq |x - y| + |x|$, so $|y| - |x| \leq |x - y|$. Now

$$-|x - y| \leq |x| - |y| \leq |x - y|,$$

giving $||x| - |y|| \leq |x - y|$. For the second inequality, we have

$$|x - y| = |x + (-y)| \leq |x| + |-y| = |x| + |y|.$$

The inequality $||x| - |y|| \leq |x - y|$ is strict iff x, y are of the opposite sign. The inequality $|x - y| \leq |x| + |y|$ is strict iff x, y are of the same sign.

Q8. Solve the inequality system

$$4 < |x + 2| + |x - 1| \leq 5.$$

Solution. We divide the real line into three parts to see if there is any solution in each part:

- **Case 1:** $x \leq -2$, then $|x+2|+|x-1| = -2-x+1-x = -1-2x$. In this case x satisfies the inequality iff $x \in (-3, -5/2]$.
- **Case 2:** $-2 < x \leq 1$, then $|x + 2| + |x - 1| = 3$. In this case there is no x satisfies the inequality.
- **Case 3:** $1 < x$, then $|x + 2| + |x - 1| = 2x + 1$. In this case x satisfies the inequality iff $x \in (3/2, 2]$.

Thus the solution set A is $[-3, -5/2) \cup (3/2, 2]$.

Q9. Let y, t be real numbers. Show the following assertions:

(a) If $|t| < 10$ and $|y - t| < 3$ then $|y| < 13$.

(b) If $t \neq 0$ and $|y - t| < |t|/2$ then $|t|/2 < |y|$.

Solution. For (a), $|y| \leq |t| + |y - t| < 3 + 10 = 13$.

For (b), $|y| \geq |t| - |y - t| > |t| - |t|/2 = |t|/2$.

Q10. Show by MI the binomial formula

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{n-k} a^{n-k} b^k.$$

And for $a > 0$, show that

$$(1+a)^n \geq 1 + na$$

and

$$(1+a)^n \geq \frac{n(n-1)(n-2)}{3!} a^2, \qquad \forall n \geq 3$$

so

$$\frac{n^2}{(1+a)^n} \to 0 \qquad \text{as } n \to \infty.$$

Solution. When $n = 0, 1$, the binomial formula is clearly true. Suppose that the binomial formula is true for $n = 0, 1, 2, \ldots, N$. Now

$$\begin{aligned}
(a+b)^{N+1} &= (a+b) \sum_{k=0}^{N} \binom{N}{N-k} a^{N-k} b^k \\
&= \sum_{k=0}^{N} \binom{N}{N-k} a^{N+1-k} b^k + \sum_{k=0}^{N} \binom{N}{N-k} a^{N-k} b^{k+1} \\
&= \binom{N}{N} a^{N+1} + \sum_{k=1}^{N} \binom{N}{N-k} a^{N+1-k} b^k + \sum_{k=0}^{N-1} \binom{N}{N-k} a^{N-k} b^{k+1} + \binom{N}{0} b^{N+1} \\
&= a^{N+1} + \sum_{k=0}^{N-1} \binom{N}{N-k-1} a^{N-k} b^{k+1} + \sum_{k=0}^{N-1} \binom{N}{N-k} a^{N-k} b^{k+1} + b^{N+1} \\
&= a^{N+1} + \sum_{k=0}^{N-1} [\binom{N}{N-k-1} + \binom{N}{N-k}] a^{N-k} b^{k+1} + b^{N+1} \\
&= a^{N+1} + \sum_{k=0}^{N-1} \binom{N+1}{N-k} a^{N-k} b^{k+1} + b^{N+1} \\
&= a^{N+1} + \sum_{k=1}^{N} \binom{N+1}{N+1-k} a^{N+1-k} b^k + b^{N+1} \\
&= \sum_{k=0}^{N+1} \binom{N+1}{N+1-k} a^{N+1-k} b^k.
\end{aligned}$$

Thus the binomial formula also holds for $n = N + 1$. By MI, the binomial formula is true for all n.

To show $(1 + a)^n \geq 1 + na$, note $1, na$ are the first two terms of the binomial expansion

$$(1 + a)^n = \sum_{k=0}^{n} \binom{n}{k} a^k.$$

Since all terms are non-negative, so $(1 + a)^n \geq 1 + na$.

The next ineq to be proved is $(1 + a)^n \geq a^2 n(n - 1)(n - 2)/3!$. The inequality should be

$$(1 + a)^n \geq \frac{n(n - 1)(n - 2)}{3!} a^3,$$

which can also be obtained from the binomial theorem.

(Fun fact: $(1 + a)^n \geq a^2 n(n - 1)(n - 2)/3!$ holds iff $n < 13$)